

# MEASURES AND INTEGRALS WITHIN CONDITIONAL SET THEORY

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**ABSTRACT.** A measure and integration theory for conditional sets is established. As an illustration of possible results, we prove a conditional version of the Carathéodory extension theorem, the monotone convergence theorem, the Fubini theorem, the Radon-Nikodým theorem, the Daniell-Stone theorem and Riesz type representation theorems. Classical function spaces in conditional set theory are defined. In order to derive the results an extension of the conditional power set as defined in Drapeau et al. is proposed. If the complete Boolean algebra is the measure algebra associated to a probability space, then it is shown that measures on the conditional Borel  $\sigma$ -algebra of a conditionally completed Polish space uniquely correspond to probability kernels. The instruments developed in this article may provide an alternative tool to study probability kernels.

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## 1. INTRODUCTION

Classical measure and integration theory is based on the fact that the power set  $\mathfrak{P}(X)$  of a set  $X$  with the operations of intersection  $\wedge = \cap$ , union  $\vee = \cup$  and complement  $^c$  and the distinguished elements  $0 = \emptyset$  and  $1 = X$  is a complete Boolean algebra. A  $\sigma$ -algebra as a plausible domain of a measurement is thus a  $\sigma$ -complete subalgebra of the Boolean algebra  $\mathfrak{P}(X)$ . For the foundations of measure theory Boolean arithmetic is indispensable. The aim of this article is to establish a measure and integration theory within conditional set theory [6]. Following the classical theory we start with a conditional form of the power set algebra. Due to the inherent locality of conditional sets more care has to be taken to specify conditional subsets.

A conditional set is a collection  $\mathbf{X}$  of objects  $x|a$  where  $x$  is an element in a non-empty set  $X$  and  $a$  is an element in a complete Boolean algebra  $\mathcal{A}$  such that a locality property

- if  $x|b = y|b$  and  $a \leq b$ , then  $x|a = y|a$ ;<sup>1</sup>

and a stability property

- if  $(a_i)$  is a partition of unity in  $\mathcal{A}$  and  $(x_i)$  is a family of elements in  $X$ , then there exists a unique element  $x$  in  $X$  such that  $x|a_i = x_i|a_i$  for all  $i$ ;

are satisfied. A conditional subset of  $\mathbf{X}$  is an object of the form  $\mathbf{C}|a$  which satisfies locally on  $a$  the stability property and is understood to be empty on  $a^c$ . A conditional measure  $\Phi$  on a collection  $\mathcal{X}$  of conditional subsets is a stable function assigning to  $\mathbf{C}|a$  locally the value 0 on  $a^c$ . Therefore the collection  $\mathcal{X}$  must satisfy some form of stability property. The stability of  $\Phi$  together with the stability of  $\mathcal{X}$  cannot be realized within the framework introduced in [6]. Therefore we suggest an extension of the conditional power set in this article. Stable collections in the extended conditional power set can serve as a domain of conditional measures if they are understood as stable functions. As in [6,

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<sup>1</sup>Recall that  $a \leq b$  is defined by  $a \wedge b = a$ .

Theorem 2.9], the extended conditional power set can be endowed with the structure of a complete Boolean algebra.

If the complete Boolean algebra  $\mathcal{A}$  is the measure algebra associated to a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , then some classical function spaces can be related to conditional spaces. For instance, the conditional real numbers are conditionally isometric with the space of real-valued random variables modulo almost sure equality.<sup>2</sup> Given a Polish space  $X$ , the space of equivalence classes of random variables with values in  $X$  can be identified with a conditional Polish space on which a conditional Borel  $\sigma$ -algebra  $\mathcal{B}$  can be defined by a stable generator of randomly open subsets in  $X$ . In Proposition 3.9, it is proved that a conditional probability measure on  $\mathcal{B}$  uniquely corresponds to a probability kernel on the Polish space  $X$ . Conditional probability measures exist on the conditional Borel  $\sigma$ -algebra of the space of equivalence classes of random variables with values in an arbitrary complete metric space. In this article, we exemplify that for measures in conditional set theory analogues of statements about measures in classical set theory can be proved. One may use conditional set theory as an alternative tool to investigate probability kernels.

We discuss the related literature. Conditional set theory is introduced in [6] and it is related to the conditional analysis in modules [1, 2, 4, 8, 9, 10, 14, 19] whenever the complete Boolean algebra is an associated measure algebra. Conditional set theory is similar to the Scott-Solovay Boolean-valued models of set theory [15, 16]. See for an investigation of classical vector lattices from the point view of Boolean-valued models [12], and the rich bibliography of [12] for further references to Boolean-valued analysis. An interpretation of differentiation and integration in Boolean-valued models is worked out in [17]. To the best of our knowledge a systematic treatment of set-theoretic measure and integration theory in conditional set theory does not exist. We refer to [5] and its references for a study of random probability measures on Polish spaces.

The rest of this paper is organized as follows. In Section 2 we fix notations and introduce the extended conditional power set and the conditional extended real numbers. Conditional set systems and set functions and a conditional version of the Carathéodory extension theorem are presented in Section 3. A conditional form of the Lebesgue integral is derived in Section 4, and there a conditional version of the monotone convergence theorem and of Fubini's theorem are proved. In Section 5 we derive a conditional version of the Radon-Nikodým theorem. Finally, in Section 6 we prove a conditional version of the Daniell-Stone theorem which is applied to Riesz representation results on the space of conditionally continuous functions on the conditionally  $n$ -dimensional Euclidean space.

## 2. NOTATION AND PRELIMINARIES

We start with basic concepts from conditional set theory as developed in [6]. We use bold-face letters to denote conditional sets and plain letters to denote classical sets. Throughout this paper we fix a complete non-degenerate Boolean algebra  $\mathcal{A} = (\mathcal{A}, \wedge, \vee, ^c, 0, 1)$  unless otherwise specified. The supremum and the infimum of a family  $(a_i) = (a_i)_{i \in I}$  of elements in  $\mathcal{A}$  are denoted by  $\vee a_i = \vee_{i \in I} a_i$  and  $\wedge a_i = \wedge_{i \in I} a_i$ , respectively. We denote by  $p(1)$  the collection of all families  $(a_i)$  of pairwise disjoint elements in  $\mathcal{A}$  such that  $\vee a_i = 1$ .

**Definition 2.1.** A collection  $\mathbf{X}$  of objects  $x|a$  where  $x$  is an element in a non-empty set  $X$  and  $a \in \mathcal{A}$  is called a *conditional set of  $X$  (and  $\mathcal{A}$ )* whenever the following properties are satisfied:

- (C1) if  $x|b = y|a$ , then  $a = b$ ;
- (C2) if  $x|b = y|b$  and  $a \leq b$ , then  $x|a = y|a$ ;
- (C3) if  $(a_i) \in p(1)$  and  $(x_i)$  is a family of elements in  $X$ , then there exists a unique element  $x$  in  $X$  such that  $x|a_i = x_i|a_i$  for all  $i$ .

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<sup>2</sup>More generally, a conditional isometry between the conditional completion of a metric space  $X$  and the space of all equivalence classes of random variables with values in  $X$  is proved in [7].

We name the properties (C2) and (C3) *consistency* and *stability*, respectively. The object  $x|a$  is referred to as  $x$  *restricted to  $a$* . A *concatenation*  $\sum x_i|a_i$  is an element  $x$  of  $X$  such that  $x|a_i = x_i|a_i$  for all  $i$  where  $(x_i)$  is family of elements in  $X$  and  $(a_i) \in p(1)$ .

We give a construction of a conditional set. Let  $X$  be a non-empty set and define  $X_s$  to be the collection of all families  $(x_i, a_i)_{i \in I} = (x_i, a_i)$  in  $X \times \mathcal{A}$  such that  $(a_i) \in p(1)$ . Identify in  $X_s$  two families  $(x_i, a_i)$  and  $(y_j, b_j)$  whenever  $\bigvee \{a_i : x_i = z\} = \bigvee \{b_j : y_j = z\}$  for all  $z \in X$ . It can be checked that  $X_s$  defines a conditional set  $\mathbf{X}_s$  of objects

$$(x_i, a_i)|a := \{(y_j, b_j) \in X_s : \bigvee \{a_i : x_i = z\} \wedge a = \bigvee \{b_j : y_j = z\} \wedge a \text{ for all } z \in X\}$$

which is called the *conditional set of step functions* with values in  $X$ . Since there exists a bijection from  $X$  to  $\{(x, 1) : x \in X\} \subseteq X_s$ , each family  $(x_i, a_i) \in X_s$  can be written as the concatenation  $\sum (x_i, 1)|a_i$ . Therefore we use the notation  $\sum x_i|a_i$  for the elements of  $X_s$ . We call the conditional sets  $\mathbf{N}_s$  and  $\mathbf{Q}_s$  of step functions with values in  $\mathbb{N}$  and  $\mathbb{Q}$  the *conditional natural* and *rational numbers*, respectively.

For a conditional version of basic concepts from measure theory families of conditional subsets of a conditional set and the conditional set operations of conditional union, conditional intersection and conditional complement are crucial. We give a detailed construction of these concepts. Let  $\mathbf{X}$  be a conditional set of  $X$ . We call a non-empty subset  $C$  of  $X$  *stable* if  $\sum x_i|a_i \in C$  for all families  $(x_i)$  in  $C$  and  $(a_i) \in p(1)$ . A stable subset  $C$  *induces* a conditional set  $\mathbf{C}$  of objects  $x|a$  where  $x \in C$  and  $a \in \mathcal{A}$ . For instance, a singleton  $\{x\}$  is a stable subset of  $X$  which induces the conditional set  $\mathbf{x} = \{x|a : a \in \mathcal{A}\}$ . A *conditional subset* of  $\mathbf{X}$  is a conditional set of the form  $\mathbf{C}|a := \{(x|a)|\tilde{a} : x|a \in C|a, \tilde{a} \in \mathcal{A}\}$  where  $C$  is a stable subset of  $X$ ,  $C|a := \{x|a : x \in C\}$  and

$$(x|a)|\tilde{a} := \{\tilde{x}|b \in \mathbf{X} : x|a \wedge \tilde{a} = \tilde{x}|b \wedge \tilde{a}, a^c \wedge \tilde{a} = b^c \wedge \tilde{a}\}.$$

Whenever there is no risk of confusion, we will write  $\mathbf{C} = \mathbf{C}|a$  for a conditional subset of  $\mathbf{X}$ . We denote by  $\mathbf{P}(\mathbf{X})$  the collection of all conditional subsets of  $\mathbf{X}$ .

Let  $\mathbf{C}|a$  and  $\mathbf{D}|b$  be conditional subsets of  $\mathbf{X}$ . We call the conditional set

$$\mathbf{C}|a \sqcap \mathbf{D}|b := \mathbf{E}|c$$

the *conditional intersection* of  $\mathbf{C}|a$  and  $\mathbf{D}|b$  where  $c = \bigvee A$ ,

$$A = \{\tilde{c} : \tilde{c} \leq a \wedge b, C|\tilde{c} \cap D|\tilde{c} \neq \emptyset\}$$

and

$$E = \{x \in X : x|c \in C|c \cap D|c\}.$$

The conditional intersection is well-defined if  $c$  is attained and  $E$  is a stable subset of  $X$ . We show that  $c$  is attained. By the axiom of choice, choose a well-ordering on the set  $A$  by means of which one defines a family  $(c_i)$  of pairwise disjoint elements in  $A$  whose supremum is equal to  $c$ . For each  $c_i$  there is an element  $x_i$  in  $X$  which restricted to  $c_i$  falls in  $C|c_i \cap D|c_i$ . The concatenation  $\sum x_i|c_i + x|c^c$  (where  $x$  may be chosen arbitrarily in  $X$ ) restricted to  $c$  is an element of  $C|c \cap D|c$ . The previous kind of reasoning is common in conditional set theory, and henceforth referred to as an *exhaustion argument*.

The conditional set

$$\mathbf{C}|a \sqcup \mathbf{D}|b := \mathbf{F}|a \vee b$$

is called the *conditional union* of  $\mathbf{C}|a$  and  $\mathbf{D}|b$  where

$$F = \{x \in X : x = z_1|a_1 + z_2|a_2 + z_3|(a \vee b)^c, (a_1, a_2, (a \vee b)^c) \in p(1), a_1 \leq a, a_2 \leq b, z_1 \in C, z_2 \in D, z_3 \in X\}$$

is a stable subset of  $X$ .

The *conditional complement* of  $\mathbf{C}|a$  is defined by

$$(\mathbf{C}|a)^\square := \mathbf{G}|d \vee a^c$$

where

$$d = \bigvee \{\tilde{d} : \tilde{d} \leq a, C|\tilde{d} \neq X|\tilde{d}\}$$

and

$$G = \{x \in X : x|\tilde{d} \neq y|\tilde{d} \forall y \in C, \forall 0 < \tilde{d} \leq d\}$$

is a stable subset of  $X$ .

The symbol  $\sqsubseteq$  stands for the *conditional inclusion*, and can be defined on  $\mathbf{P}(\mathbf{X})$  by  $\mathbf{C}|a \sqsubseteq \mathbf{D}|b$  whenever  $\mathbf{C}|a \cap \mathbf{D}|b = \mathbf{C}|a$ .<sup>3</sup> We recall [6, Theorem 2.9].

**Theorem 2.2.** *The structure  $(\mathbf{P}(\mathbf{X}), \cap, \sqcup, \sqsubseteq, \mathbf{X}|0, \mathbf{X})$  is a complete Boolean algebra.*

We introduce the extended conditional power set. Given  $\mathbf{C}|a \in \mathbf{P}(\mathbf{X})$  and  $\tilde{a} \in \mathcal{A}$ , we define

$$(\mathbf{C}|a)|\tilde{a} := \{\mathbf{D}|b \in \mathbf{P}(\mathbf{X}) : \mathbf{C}|a \wedge \tilde{a} = \mathbf{D}|b \wedge \tilde{a}, a^c \wedge \tilde{a} = b^c \wedge \tilde{a}\}.$$

It can be checked that

$$\mathcal{P}(\mathbf{X}) := \{(\mathbf{C}|a)|\tilde{a} : \mathbf{C}|a \in \mathbf{P}(\mathbf{X}), \tilde{a} \in \mathcal{A}\}$$

is a conditional set of  $\mathbf{P}(\mathbf{X})$  which we call the *conditional power set* of  $\mathbf{X}$ . Throughout, a *stable collection of conditional subsets* of  $\mathbf{X}$  is a stable subset of  $\mathbf{P}(\mathbf{X})$  with respect to concatenations formed in  $\mathcal{P}(\mathbf{X})$ .

In the following paragraphs let  $\mathbf{X}$  be a conditional set of  $X$  and  $\mathbf{Y}$  be a conditional set of  $Y$ . The *conditional product* of  $\mathbf{X}$  and  $\mathbf{Y}$  is defined as the conditional set

$$\mathbf{X} \times \mathbf{Y} := \{(x, y)|a := (x|a, y|a) : (x, y) \in X \times Y, a \in \mathcal{A}\}.$$

Let  $\mathbf{W}|a$  be a conditional subset of  $\mathbf{X} \times \mathbf{Y}$  and  $x \in X$ . We define the *conditional  $x$ -section* of  $\mathbf{W}|a$  by

$$(\mathbf{W}|a)_x := \mathbf{C}|b$$

where

$$b = \vee \left\{ \tilde{b} : \tilde{b} \leq a, \exists y \in Y, \exists (\tilde{x}, \tilde{y}) \in W \text{ such that } (x, y)|\tilde{b} = (\tilde{x}, \tilde{y})|\tilde{b} \right\}$$

and

$$C = \{y \in Y : \exists (\tilde{x}, \tilde{y}) \in W \text{ such that } (x, y)|b = (\tilde{x}, \tilde{y})|b\}.$$

By an exhaustion argument,  $b$  is attained, and thus it can be verified that  $C$  is a stable subset of  $Y$ . Similarly the *conditional  $y$ -section* of  $\mathbf{W}|a$  for an element  $y$  in  $Y$  is defined.

We recall the definition of a stable function and family. A function  $f : X \rightarrow Y$  is called *stable* if  $f(\sum x_i|a_i) = \sum f(x_i)|a_i$  for all concatenations  $\sum x_i|a_i$  in  $X$ . Let  $\mathbf{C}|a$  be a conditional subset of  $\mathbf{Y}$ . We define the *conditional pre-image* by  $f^{-1}(\mathbf{C}|a) := \mathbf{D}|b$  where

$$b = \vee \left\{ \tilde{b} : \tilde{b} \leq a, \exists x \in X, \exists y \in C \text{ such that } f(x)|\tilde{b} = y|\tilde{b} \right\}$$

and

$$D = \{x \in X : \exists y \in C \text{ such that } f(x)|b = y|b\}.$$

We define a stable order<sup>4</sup> on  $\mathbb{N}_s$  by  $\sum n_i|a_i \leq \sum m_j|b_j$  if  $n_i \leq m_j$  for all  $a_i \wedge b_j > 0$ . Note that  $\{1 \leq m \leq n\}$  is a stable subset of  $\mathbb{N}_s$  for every  $n \in \mathbb{N}_s$ . A *stably finite family*  $(x_m)_{1 \leq m \leq n}$  in  $X$  is the graph of a stable function  $\{1 \leq m \leq n\} \rightarrow X$ . A *stable sequence*  $(x_n)$  in  $X$  is the graph of a stable function  $\mathbb{N}_s \rightarrow X$ .

The following properties follow from the previous definitions. Let  $(\mathbf{C}_i) = (\mathbf{C}_i|a_i)$  be a non-empty family of conditional subsets of  $\mathbf{X}$ ,  $\mathbf{D} \sqsubseteq \mathbf{X}$  and  $(b_i) \in p(1)$ . Then one has

- (S1)  $\cap(\mathbf{C}_i|a_i) = (\cap \mathbf{C}_i)|a_i$  and  $\sqcup(\mathbf{C}_i|a_i) = (\sqcup \mathbf{C}_i)|a_i$ ,
- (S2)  $(\sum(\mathbf{C}_i|a_i)|b_i)^\sqsubseteq = \sum(\mathbf{C}_i|a_i)^\sqsubseteq|b_i + \mathbf{X} \vee (a_i^c \wedge b_i)$ ,
- (S3)  $(\sum \mathbf{C}_i|b_i) \cap \mathbf{D} = \sum(\mathbf{C}_i \cap \mathbf{D})|b_i$  and  $(\sum \mathbf{C}_i|b_i) \sqcup \mathbf{D} = \sum(\mathbf{C}_i \sqcup \mathbf{D})|b_i$ .

<sup>3</sup>For a general definition we refer to [6, Definition 2.8].

<sup>4</sup>See [6, Definition 2.15].

For a non-empty family  $(\mathbf{C}_{ij})$  of conditional subsets of  $\mathbf{X}$ , a stably finite family  $(\mathbf{C}_m)_{1 \leq m \leq n}$  of conditional subsets of  $\mathbf{X}$  and  $(b_i) \in p(1)$ , one has

$$\begin{aligned} \text{(S4)} \quad & \sum (\sqcup_j \mathbf{C}_{ij}) | b_i = \sqcup_j (\sum \mathbf{C}_{ij} | b_i) \text{ and } \sum (\cap_j \mathbf{C}_{ij}) | b_i = \cap_j (\sum \mathbf{C}_{ij} | b_i), \\ \text{(S5)} \quad & \cap_{1 \leq m \leq n} \mathbf{C}_m = \sum (\cap_{1 \leq m \leq n_i} \mathbf{C}_m) | a_i \text{ and } \sqcup_{1 \leq m \leq n} \mathbf{C}_m = \sum (\sqcup_{1 \leq m \leq n_i} \mathbf{C}_m) | a_i \text{ where } n = \sum n_i | a_i. \end{aligned}$$

Let  $\mathbf{C} \times \mathbf{D} = \mathbf{C} | a \times \mathbf{D} | b$  and  $\tilde{\mathbf{C}} \times \tilde{\mathbf{D}}$  be conditional subsets of  $\mathbf{X} \times \mathbf{Y}$ ,  $(\mathbf{C}^i)$  be a family of conditional subsets of  $\mathbf{X} \times \mathbf{Y}$ ,  $\mathbf{E} \subseteq \mathbf{X} \times \mathbf{Y}$  and  $(b_i) \in p(1)$ . Then we have

$$\begin{aligned} \text{(S6)} \quad & \mathbf{C} \times \mathbf{D} \cap \tilde{\mathbf{C}} \times \tilde{\mathbf{D}} = \mathbf{C} \cap \tilde{\mathbf{C}} \times \mathbf{D} \cap \tilde{\mathbf{D}}, \\ \text{(S7)} \quad & (\sum \mathbf{C}^i | b_i)_x = \sum \mathbf{C}_x^i | b_i \text{ for all } x \in X, \\ \text{(S8)} \quad & (\mathbf{E} | a)_x = \mathbf{E}_x | a \text{ for all } x \in X \text{ and } a \in \mathcal{A}, \\ \text{(S9)} \quad & \mathbf{C}_{\sum x_i | a_i} = \sum \mathbf{C}_{x_i} | a_i \text{ for all concatenations } \sum x_i | a_i \text{ in } X, \\ \text{(S10)} \quad & (\mathbf{E}_x)^\square = (\mathbf{E}^\square)_x \text{ for all } x \in X, \\ \text{(S11)} \quad & (\sqcup \mathbf{C}^i)_x = \sqcup \mathbf{C}_x^i \text{ for all } x \in X, \\ \text{(S12)} \quad & (\mathbf{C} | a \times \mathbf{D} | b)_x = \mathbf{D} | c \text{ where } c = \vee \{ \tilde{c} : \tilde{c} \leq a, \exists \tilde{x} \in C \text{ such that } x | \tilde{c} = \tilde{x} | \tilde{c} \} \text{ for all } x \in X. \end{aligned}$$

We close this preliminary section with basic properties of the conditional extended real line. The *conditional real numbers*  $\mathbf{R}$  are constructed in [6, Section 4]. The *conditional extended real numbers*  $\overline{\mathbf{R}}$  can be obtained by a conditional completion<sup>5</sup> of the conditional set  $\overline{\mathbf{R}}_s$  of step functions with values in  $\overline{\mathbf{R}} := \mathbf{R} \cup \{\pm\infty\}$ . Denoting by  $R$  and  $\overline{R}$  the stable sets inducing  $\mathbf{R}$  and  $\overline{\mathbf{R}}$ , respectively, each  $x$  in  $\overline{R}$  has the form  $-\infty | a_1 + y | a_2 + \infty | a_3$  where  $y \in R$  and  $(a_1, a_2, a_3) \in p(1)$ . The conditional addition and multiplication on  $\mathbf{R}$  extend to  $\overline{\mathbf{R}}$  by stipulating the following conditional version of the classical conventions. For all  $-\infty | a_1 + x | a_2 + \infty | a_3, -\infty | b_1 + y | b_2 + \infty | b_3 \in \overline{R}$  and  $r = r^- | c_1 + 0 | c_2 + r^+ | c_3 \in R$  where  $c_1 = \vee \{a : r | a < 0 | a\}$ ,  $c_2 = \vee \{a : r | a = 0 | a\}$  and  $c_3 = \vee \{a : r | a > 0 | a\}$ , and  $r^- = r | c_1 + 0 | c_1^c$  and  $r^+ = r | c_3 + 0 | c_3^c$ , set

$$\begin{aligned} (-\infty | a_1 + x | a_2 + \infty | a_3) + (-\infty | b_1 + y | b_2 + \infty | b_3) &:= -\infty | (a_1 \wedge b_1) \vee (a_1 \wedge b_2) \vee (b_1 \wedge a_2) \\ &\quad + 0 | (a_1 \wedge b_3) \vee (a_3 \wedge b_1) + (x + y) | (a_2 \wedge b_2) \\ &\quad + \infty | (a_3 \wedge b_3) \vee (a_2 \wedge b_3) \vee (b_2 \wedge a_3), \\ r \cdot (-\infty | a_1 + x | a_2 + \infty | a_3) &:= -\infty | (c_3 \wedge a_1) \vee (c_1 \wedge a_3) \\ &\quad + r^- x | c_1 \wedge a_2 + 0 | c_2 + r^+ x | c_3 \wedge a_2 \\ &\quad + \infty | (c_3 \wedge a_3) \vee (c_1 \wedge a_1). \end{aligned}$$

The total order<sup>6</sup> on  $R$  extends to  $\overline{R}$  via the relation

$$-\infty | a_1 + x | a_2 + \infty | a_3 \leq -\infty | b_1 + y | b_2 + \infty | b_3 \quad \text{if} \quad b_1 \leq a_1, a_3 \leq b_3 \text{ and } x | a_2 \wedge b_2 \leq y | a_2 \wedge b_2.$$

Let  $\overline{R}_+ := \{x \in \overline{R} : x \geq 0\}$  and  $R_{++} := \{x \in R : x > 0\}$  where the relation  $x > 0$  is defined as  $x \geq 0$  and  $x | a = 0 | a$  whenever  $a = 0$ . Note that  $\overline{R}_+$  and  $R_{++}$  are stable subsets of  $\overline{R}$  and  $R$ , respectively. For  $n = \sum n_i | a_i \in \mathbb{N}_s$  and a stably finite family  $(x_m)_{1 \leq m \leq n}$  in  $\overline{R}$ , define the *stably finite sum* and *stably finite product* by

$$\sum_{1 \leq m \leq n} x_m := \sum (x_1 + \dots + x_{n_i}) | a_i, \quad \prod_{1 \leq m \leq n} x_m := \sum (x_1 \cdot \dots \cdot x_{n_i}) | a_i,$$

respectively. The collection of all stable open balls<sup>7</sup>  $B_r(x) := \{y \in \overline{R} : d_c(x, y) < r\}$  where  $x \in \overline{R}$  and  $r \in \overline{R}_+$  is a stable collection of stable subsets which forms a classical topological base on the set  $\overline{R}$ , and thus induces a conditional topology<sup>8</sup>  $\mathcal{T}$  on  $\overline{\mathbf{R}}$  by [6, Proposition 3.5]. Restricted to  $\mathbf{R}$ , the conditional topology  $\mathcal{T}$  is conditionally equivalent to the conditional Euclidean topology which is defined in [6,

<sup>5</sup>See [7, Theorem 2.1].

<sup>6</sup>See [6, Definition 2.15].

<sup>7</sup> $d_c$  is a stable metric with values in  $R$  induced by the arctan-metric on  $\overline{\mathbf{R}}$ , see [7, Section 2].

<sup>8</sup>See [6, Definition 3.1].

Section 4]. For a stable sequence  $(x_n)$  in  $\overline{R}_+$ , we define the *stably infinite series*  $\sum_{n \geq 1} x_n$  as the limit of the stable sequence of stably finite sums  $\sum_{1 \leq m \leq n} x_m$ .

### 3. CONDITIONAL MEASURE SPACES AND A CONDITIONAL VERSION OF CARATHÉODORY'S EXTENSION THEOREM

In this section we introduce a conditional version of measurability. We prove a conditional version of Carathéodory's extension theorem by means of which we construct a conditional  $n$ -dimensional Lebesgue measure.

**Definition 3.1.** Let  $\mathbf{X}$  be a conditional set. A stable collection  $\mathcal{X}$  of conditional subsets of  $\mathbf{X}$  is called

- a *conditional ring* whenever  $\mathbf{C} \sqcap \mathbf{D}^\square, \mathbf{C} \sqcup \mathbf{D} \in \mathcal{X}$  for all  $\mathbf{C}, \mathbf{D} \in \mathcal{X}$ ;
- a *conditional Dynkin system* whenever  $\mathbf{X} \in \mathcal{X}$ ,  $\mathbf{C}^\square \in \mathcal{X}$  for all  $\mathbf{C} \in \mathcal{X}$  and  $\sqcup \mathbf{C}_n \in \mathcal{X}$  for all stable sequences  $(\mathbf{C}_n)$  of conditionally pairwise disjoint<sup>9</sup> conditional sets in  $\mathcal{X}$ ;
- a *conditional  $\sigma$ -algebra* whenever  $\mathbf{X} \in \mathcal{X}$ ,  $\mathbf{C}^\square \in \mathcal{X}$  for all  $\mathbf{C} \in \mathcal{X}$  and  $\sqcup \mathbf{C}_n \in \mathcal{X}$  for all stable sequences  $(\mathbf{C}_n)$  of conditional sets in  $\mathcal{X}$ .

A pair  $(\mathbf{X}, \mathcal{X})$  is called a *conditional measurable space* if  $\mathbf{X}$  is a conditional set and  $\mathcal{X}$  is a conditional  $\sigma$ -algebra. Let  $(\mathbf{X}, \mathcal{X})$  and  $(\mathbf{Y}, \mathcal{Y})$  be conditional measurable spaces. A stable function  $f: X \rightarrow Y$  is called *conditionally measurable* if  $f^{-1}(\mathbf{C}) \in \mathcal{X}$  for all  $\mathbf{C} \in \mathcal{Y}$ .

For a stable collection  $\mathcal{E}$  of conditional subsets of  $\mathbf{X}$ , it can be checked that

$$\begin{aligned} \Sigma(\mathcal{E}) &:= \cap \{ \mathcal{X} : \mathcal{E} \subseteq \mathcal{X}, \mathcal{X} \text{ conditional } \sigma\text{-algebra} \}, \\ \Delta(\mathcal{E}) &:= \cap \{ \mathcal{X} : \mathcal{E} \subseteq \mathcal{X}, \mathcal{X} \text{ conditional Dynkin system} \} \end{aligned}$$

are a conditional  $\sigma$ -algebra and Dynkin system called the conditional  $\sigma$ -algebra and Dynkin system generated by  $\mathcal{E}$ , respectively. If  $\mathcal{E}$  is a conditional topology on  $\mathbf{X}$ , then the conditional  $\sigma$ -algebra  $\Sigma(\mathcal{E})$  is called the *conditional Borel  $\sigma$ -algebra* of  $\mathbf{X}$ .<sup>10</sup> We have the following conditional version of Dynkin's  $\pi$ - $\lambda$  theorem.

**Theorem 3.2.** Let  $\mathbf{X}$  be a conditional set and  $\mathcal{E}$  be a stable collection of conditional subsets of  $\mathbf{X}$  which is closed under stably finite conditional intersections. Then one has

$$\Sigma(\mathcal{E}) = \Delta(\mathcal{E}).$$

*Proof.* The proof is similar to the classical proof by applying Theorem 2.2. □

**Definition 3.3.** Let  $\mathbf{X}$  be a conditional set and  $\mathcal{X}$  a conditional ring. A stable function  $\Phi: \mathcal{X} \rightarrow \overline{R}_+$  is called a *conditional pre-measure* whenever

- (M1)  $\Phi(\mathbf{C}|a) = \Phi(\mathbf{C})|a + 0|a^c$  for all  $\mathbf{C}$  in  $\mathcal{X}$ ,  $a \in \mathcal{A}$ ;
- (M2)  $\Phi(\sqcup \mathbf{C}_n) = \sum_{n \geq 1} \Phi(\mathbf{C}_n)$  for all stable sequences  $(\mathbf{C}_n)$  of conditionally pairwise disjoint conditional sets in  $\mathcal{X}$  with  $\sqcup \mathbf{C}_n \in \mathcal{X}$ .

A stable function  $\Phi: \mathcal{X} \rightarrow \overline{R}_+$  satisfying property (M1) and

<sup>9</sup>A stable sequence  $(\mathbf{C}_n)$  is *conditionally pairwise disjoint* if  $\mathbf{C}_n \sqcap \mathbf{C}_m = \mathbf{X}|0$  whenever  $n$  is conditionally not equal to  $m$ , i.e.  $n|a = m|a$  implies  $a = 0$ .

<sup>10</sup>A conditional topology is defined in [6, Definition 3.1] to be a conditional collection of conditional subsets in the conditional set  $\mathbf{P}(\mathbf{X})$  (see [6, Definition 2.7]) which in the extended conditional power set  $\mathcal{P}(\mathbf{X})$  can be seen as a stable collection of conditional subsets.

(M2)'  $\Phi(\sqcup_{1 \leq m \leq n} \mathbf{C}_m) = \sum_{1 \leq m \leq n} \Phi(\mathbf{C}_m)$  for all stably finite families  $(\mathbf{C}_m)_{1 \leq m \leq n}$  of conditionally pairwise conditional sets in  $\mathcal{X}$ ,

is called a *conditional content*. If the domain of a conditional pre-measure is a conditional  $\sigma$ -algebra, then it is called a *conditional measure* and the triple  $(\mathbf{X}, \mathcal{X}, \Phi)$  a *conditional measure space*. A conditional measure  $\Phi$  is *conditionally finite* if  $\Phi(\mathbf{X}) < \infty$ , and *conditionally  $\sigma$ -finite* if there exists a stable sequence  $(\mathbf{D}_n)$  in  $\mathcal{X}$  with  $\mathbf{D}_m \subseteq \mathbf{D}_n$  whenever  $m \leq n$  and  $\sqcup \mathbf{D}_n = \mathbf{X}$  such that  $\Phi(\mathbf{D}_n) < \infty$  for all  $n$ .

Let  $\Phi$  be a conditional content on  $(\mathbf{X}, \mathcal{X})$ . By inspection, one has

- (M3)  $\Phi(\mathbf{C} \sqcup \mathbf{D}) + \Phi(\mathbf{C} \cap \mathbf{D}) = \Phi(\mathbf{C}) + \Phi(\mathbf{D})$  for all  $\mathbf{C}, \mathbf{D}$  in  $\mathcal{X}$ ;
- (M4)  $\Phi(\mathbf{C}) \leq \Phi(\mathbf{D})$  for all  $\mathbf{C}, \mathbf{D}$  in  $\mathcal{X}$  with  $\mathbf{C} \subseteq \mathbf{D}$ ;
- (M5)  $\Phi(\mathbf{D} \cap \mathbf{C}^c) = \Phi(\mathbf{D}) - \Phi(\mathbf{C})$  for all  $\mathbf{C}, \mathbf{D}$  in  $\mathcal{X}$  with  $\mathbf{C} \subseteq \mathbf{D}$  and  $\Phi(\mathbf{C}) < \infty$ ;
- (M6)  $\Phi(\sqcup_{n \geq 1} \mathbf{C}_n) \leq \sum_{n \geq 1} \Phi(\mathbf{C}_n)$  for all stable sequences  $(\mathbf{C}_n)$  of conditional sets in  $\mathcal{X}$  with  $\sqcup_{n \geq 1} \mathbf{C}_n \in \mathcal{X}$ ;
- (M7)  $\Phi$  is a conditional pre-measure if and only if for all stable sequences  $(\mathbf{C}_n)$  of conditional sets in  $\mathcal{X}$  such that  $\mathbf{C}_m \subseteq \mathbf{C}_n$  whenever  $m \leq n$  and  $\sqcup \mathbf{C}_n = \mathbf{C}$  for some  $\mathbf{C}$  in  $\mathcal{X}$ , one has  $\lim \Phi(\mathbf{C}_n) = \Phi(\mathbf{C})$ ;
- (M8) If  $\Phi(\mathbf{C}) < \infty$  for all  $\mathbf{C}$  in  $\mathcal{X}$ , then  $\Phi$  is a conditional pre-measure if and only if for all stable sequences  $(\mathbf{C}_n)$  of conditional sets in  $\mathcal{X}$  such that  $\mathbf{C}_n \subseteq \mathbf{C}_m$  whenever  $m \leq n$  and  $\cap \mathbf{C}_n = \mathbf{C}$  for some  $\mathbf{C}$  in  $\mathcal{X}$ , one has  $\lim \Phi(\mathbf{C}_n) = \Phi(\mathbf{C})$ .

**Definition 3.4.** Given a conditional set  $\mathbf{X}$ , a stable function  $\Phi^*: \mathbf{P}(\mathbf{X}) \rightarrow \overline{\mathbf{R}}_+$  is called a *conditional outer measure* whenever

- (A1)  $\Phi^*(\mathbf{C}|a) = \Phi^*(\mathbf{C})|a + 0|a^c$  for all  $\mathbf{C}$  in  $\mathbf{P}(\mathbf{X})$ ,  $a \in \mathcal{A}$ ;
- (A2)  $\Phi^*(\mathbf{C}) \leq \Phi^*(\mathbf{D})$  for all  $\mathbf{C}, \mathbf{D}$  in  $\mathbf{P}(\mathbf{X})$  with  $\mathbf{C} \subseteq \mathbf{D}$ ;
- (A3)  $\Phi^*(\sqcup_{n \geq 1} \mathbf{C}_n) \leq \sum_{n \geq 1} \Phi^*(\mathbf{C}_n)$  for all stable sequences  $(\mathbf{C}_n)$  in  $\mathbf{P}(\mathbf{X})$ .

A conditional subset  $\mathbf{C}$  of  $\mathbf{X}$  is said to be *conditionally  $\Phi^*$ -measurable* whenever

$$\Phi^*(\mathbf{D} \cap \mathbf{C}) + \Phi^*(\mathbf{D} \cap \mathbf{C}^c) = \Phi^*(\mathbf{D})$$

for all conditional subsets  $\mathbf{D}$  of  $\mathbf{X}$ . Let  $\mathcal{X}(\Phi^*)$  denote the stable collection of all conditionally  $\Phi^*$ -measurable conditional sets.

Similarly to the classical case, one can prove that  $\mathcal{X}(\Phi^*)$  is a conditional  $\sigma$ -algebra and  $\Phi^*$  restricted to  $\mathcal{X}(\Phi^*)$  is a conditional measure. This provides for the following conditional version of Carathéodory's extension theorem.

**Theorem 3.5.** Let  $\mathbf{X}$  be a conditional set,  $\mathcal{X}$  a conditional ring on  $\mathbf{X}$  and  $\Phi: \mathcal{X} \rightarrow \overline{\mathbf{R}}_+$  a conditional pre-measure. Then there exists a conditional measure  $\Psi: \Sigma(\mathcal{X}) \rightarrow \overline{\mathbf{R}}_+$  which coincides with  $\Phi$  on  $\mathcal{X}$ .

*Proof.* For a conditional subset  $\mathbf{C}$  of  $\mathbf{X}$ , put

$$a_{\mathbf{C}} := \vee \{a: \exists \text{ a stable sequence } (\mathbf{D}_n) \text{ in } \mathcal{X} \text{ such that } \mathbf{C}|a \subseteq \sqcup \mathbf{D}_n\}.$$

By an exhaustion argument,  $a_{\mathbf{C}}$  is attained. Thus  $\Phi^*: \mathbf{P}(\mathbf{X}) \rightarrow \overline{\mathbf{R}}_+$  given by

$$\Phi^*(\mathbf{C}) := \inf \left\{ \sum_{n \geq 1} \Phi(\mathbf{D}_n): (\mathbf{D}_n) \text{ stable sequence in } \mathcal{X} \text{ such that } \mathbf{C}|a_{\mathbf{C}} \subseteq \sqcup \mathbf{D}_n \right\} \Big| a_{\mathbf{C}} + \infty | a_{\mathbf{C}}^c.$$

is a well-defined stable function. It can be checked, similarly to the classical case, that  $\Phi^*$  is a conditional outer measure.  $\square$

As for the uniqueness of the extension in the previous theorem, one has the following result which can be proved by using Theorem 3.2.

**Proposition 3.6.** *Let  $(\mathbf{X}, \mathcal{X}, \Phi)$  and  $(\mathbf{X}, \mathcal{X}, \Psi)$  be conditional measure spaces,  $\mathcal{E}$  a stable collection of conditional subsets of  $\mathcal{X}$  which is closed under stably finite conditional intersections and such that  $\mathcal{X} = \Sigma(\mathcal{E})$ . Further there exists a stable sequence  $(\mathbf{D}_n)$  in  $\mathcal{E}$  with  $\sqcup \mathbf{D}_n = \mathbf{X}$  such that  $\Phi(\mathbf{D}_n) = \Psi(\mathbf{D}_n) < \infty$  for all  $n$ . If one has  $\Phi(\mathbf{C}) = \Psi(\mathbf{C})$  for all  $\mathbf{C} \in \mathcal{E}$ , then we have  $\Phi(\mathbf{C}) = \Psi(\mathbf{C})$  for all  $\mathbf{C} \in \mathcal{X}$ .*

We give two examples of conditional measures.

**Examples 3.7.** (i) Let  $\mathbf{X}$  be a conditional set of  $X$ ,  $\mathcal{X}$  a conditional  $\sigma$ -algebra on  $\mathbf{X}$  and  $x \in X$ . The conditional Dirac measure centered at  $x$  is defined as

$$\delta_x(\mathbf{C}) := 1|a + 0|a^c, \quad \mathbf{C} \text{ in } \mathcal{X},$$

where  $a = \vee \{\tilde{a} : \exists \tilde{x} \in C \text{ such that } x|\tilde{a} = \tilde{x}|\tilde{a}\}$ .

(ii) For  $n = \sum n_i | a_i \in \mathbb{N}_s$ , we denote by  $\mathbf{R}^n$  the conditionally  $n$ -dimensional Euclidean space. An element  $x$  in the stable set  $R^n$  has the form  $x = (x_m)_{1 \leq m \leq n} := \sum (x_{i1}, \dots, x_{in_i}) | a_i$  with  $(x_1, \dots, x_{n_i})$  being an element of the product  $R^{n_i}$ , each  $i$ . For  $x, y$  in  $R^n$  let  $x \leq y$  be defined by  $x_{im} \leq y_{im}$  for all  $1 \leq m \leq n_i$  and  $i$ . Given  $x \leq y$ , we define the stable subset  $[x, y] := \{z \in R^n : x \leq z \leq y\}$ . It can be checked that

$$\mathcal{X} = \{\sqcup_{1 \leq l \leq k} [\mathbf{x}_l, \mathbf{y}_l] | a : ([\mathbf{x}_l, \mathbf{y}_l])_{1 \leq l \leq k} \text{ stably finite, conditionally pairwise disjoint, } a \in \mathcal{A}\}$$

is a conditional ring. By a conditional version of the Heine-Borel theorem [6, Theorem 4.6] and (M8), the function  $\Lambda : \mathcal{X} \rightarrow \overline{R}_+$  given by

$$\Lambda(\sqcup_{1 \leq l \leq k} [\mathbf{x}_l, \mathbf{y}_l] | a) := \sum_{1 \leq l \leq k} \prod_{1 \leq m \leq n} (y_{lm} - x_{lm}) | a + 0 | a^c$$

is a conditional pre-measure. By Theorem 3.5 and Proposition 3.6,  $\Lambda$  uniquely extends to a conditional measure on the conditional Borel  $\sigma$ -algebra  $\mathcal{B}^n := \Sigma(\mathcal{X})$  which is called the *conditional  $n$ -dimensional Lebesgue measure*.

In the following we will give a canonical construction of conditional measures induced by probability kernels on Polish spaces where the complete Boolean algebra is associated to a  $\sigma$ -finite measure space.

**3.8. Probability kernels and conditional measures.** For the remaining part of this section  $\mathcal{A}$  denotes the measure algebra associated to a fixed complete  $\sigma$ -finite measure space  $(\Omega, \mathcal{F}, \mu)$ . Recall that  $\mathcal{A}$  is obtained from  $\mathcal{F}$  by quotienting out the  $\sigma$ -ideal of  $\mu$ -null sets. By [13, Chapter 22, Proposition 2.8], the Boolean algebra  $\mathcal{A}$  is  $\sigma$ -complete and satisfies the countable chain condition, and thus is complete [18]. We denote the equivalence classes in  $\mathcal{A}$  by  $a = [A]$  where  $A \in \mathcal{F}$ . We write  $1_A$  for the characteristic function of  $A \in \mathcal{F}$ . Let  $\overline{L}^0$  be the collection of all measurable functions  $x : \Omega \rightarrow \mathbb{R}$  where two of them are identified if they agree  $\mu$ -almost everywhere, and denote by  $L^0$  the subset of  $\overline{L}^0$  consisting of finite-valued functions. Equalities and inequalities between measurable functions are understood in the  $\mu$ -almost everywhere sense. The set  $\overline{L}^0$  induces a conditional set  $\overline{\mathbf{L}}^0$  of objects

$$x|a := \{y \in \overline{L}^0 : x1_A = y1_A \text{ for some } A \in a\} \quad (3.1)$$

where  $x \in \overline{L}^0$  and  $a \in \mathcal{A}$ . We denote by  $\mathbf{L}^0$  the conditional subset of  $\overline{\mathbf{L}}^0$  induced by the stable subset  $L^0$ . By the countable chain condition, the elements in  $p(1)$  have at most countably many non-zero entries. For  $(a_n) \in p(1)$  and  $(x_n)$  a sequence of elements in  $\overline{L}^0$  the concatenation  $\sum x_n | a_n$  is the measurable function  $\sum_{n \geq 1} x_n 1_{A_n}$  where  $a_n = [A_n]$  each  $n$ . By adapting the proof of [6, Theorem 4.4], one shows that  $\overline{\mathbf{L}}^0$  is conditionally isometrically isomorphic to  $\overline{\mathbf{R}}$ . The stable order on  $\overline{\mathbf{R}}$  corresponds to the order of almost everywhere dominance on  $\overline{L}^0$ . We identify  $\overline{R}_+$ ,  $\mathbb{N}_s$  and  $\mathbb{Q}_s$  with the stable sets of all  $\overline{\mathbf{R}}_+$ ,  $\mathbf{N}$ - and  $\mathbf{Q}$ -valued measurable functions, respectively.



Recall that given a Polish space  $S$  with its Borel  $\sigma$ -algebra  $\mathcal{S}$ , a probability kernel from  $(\Omega, \mathcal{F})$  to  $(S, \mathcal{S})$  is a function  $\kappa: \Omega \times \mathcal{S} \rightarrow \overline{\mathbb{R}}_+$  such that  $\kappa(\omega, B)$  is measurable in  $\omega$  for fixed  $B \in \mathcal{S}$  and a probability measure in  $B$  for fixed  $\omega \in \Omega$ . The following result relates probability kernels with conditional probability measures on the conditional Borel  $\sigma$ -algebra of the conditional completion<sup>11</sup>  $\mathbf{S}_c$ . Denoting by  $L^0(S)$  the space of all Borel measurable functions  $x: \Omega \rightarrow S$  modulo  $\mu$ -almost everywhere equality, by a copy of the proof of [7, Theorem 4.1],  $\mathbf{S}_c$  is conditionally isometrically isomorphic to the conditional set  $\mathbf{L}^0(\mathbf{S})$  where  $\mathbf{L}^0(\mathbf{S})$  is defined as in (3.1). By [11, Theorem A1.2], there exists a Borel isomorphism from  $S$  to a Borel subset of  $\mathbb{R}$ . Therefore it is enough to prove the following assertion where  $\mathfrak{B}$  denotes the Borel  $\sigma$ -algebra of  $\mathbb{R}$  and  $\mathcal{B} = \mathcal{B}^1$  the conditional Borel  $\sigma$ -algebra of  $\mathbf{L}^0$ .

**Proposition 3.9.** *To every probability kernel  $\kappa: \Omega \times \mathfrak{B} \rightarrow \overline{\mathbb{R}}_+$  is uniquely associated a conditional probability measure  $\Phi_\kappa: \mathcal{B} \rightarrow \overline{\mathbb{L}}_+^0$  and to every conditional probability measure  $\Phi: \mathcal{B} \rightarrow \overline{\mathbb{L}}_+^0$  is uniquely associated a probability kernel  $\kappa_\Phi: \Omega \times \mathfrak{B} \rightarrow \overline{\mathbb{R}}_+$  such that the identities  $\Phi_{\kappa_\Phi} = \Phi$  and  $\kappa_{\Phi_\kappa} = \kappa$  hold almost everywhere.*

*Proof.* Let  $\kappa: \Omega \times \mathfrak{B} \rightarrow \overline{\mathbb{R}}_+$  be a probability kernel. For  $x, y \in \mathbb{Q}_s$  with  $x \leq y$  there exists  $(a_n) \in p(1)$  and a sequence  $(x_n, y_n)$  in  $\mathbb{Q}^2$  with  $x_n \leq y_n$  each  $n$  such that  $[x, y] = \sum_{n \geq 1} [x_n, y_n] 1_{A_n}$  where  $a_n = [A_n]$  each  $n$ . On

$\mathcal{X} = \{\sqcup_{1 \leq l \leq k} [\mathbf{x}_l, \mathbf{y}_l] | a: ([\mathbf{x}_l, \mathbf{y}_l])_{1 \leq l \leq k} \text{ stably finite, conditionally pairwise disjoint, } x_l, y_l \in \mathbb{Q}_s, a \in \mathcal{A}\}$  define  $\Phi_\kappa$  by

$$\Phi_\kappa(\sqcup_{1 \leq l \leq k} [\mathbf{x}_l, \mathbf{y}_l] | a) := \sum_{n \geq 1} \left( \sum_{1 \leq l \leq k_n} \left( \sum_{m \geq 1} \kappa(\omega, [x_{lm}, y_{lm}]) 1_{A_{lm}}(\omega) \right) \right) 1_{A_n \cap A}$$

where  $k = \sum_{n \geq 1} k_n 1_{A_n}$  and  $[x_l, y_l] = \sum_{m \geq 1} [x_{lm}, y_{lm}] 1_{A_{lm}}$ . By Theorem 3.5 and Proposition 3.6,  $\Phi_\kappa$  uniquely extends to a conditional probability measure on  $\mathcal{B}$ . Conversely, let  $\Phi: \mathcal{B} \rightarrow \overline{\mathbb{L}}_+^0$  be a conditional probability measure. For each  $q \in \mathbb{Q}$  define  $f(\cdot, q): \Omega \rightarrow [0, 1]$  by  $f(\omega, q) = \Phi((-\infty, q])(\omega)$ . By (M4), one has  $f(\cdot, p) \leq f(\cdot, q)$  whenever  $p \leq q$ . Therefore  $A = \{\omega \in \Omega: f(\omega, q) \text{ increasing in } q \in \mathbb{Q}\}$  is measurable. For each  $\omega \in A$  the function  $f(\omega, q)$  is increasing in  $q$  with limits 0 and 1 at  $-\infty$  and  $+\infty$ , respectively. Set

$$F(\omega, x) := 1_A(\omega) \inf_{q > x} f(\omega, q) + 1_{A^c}(\omega) 1_{\{x \geq 0\}}, \quad x \in \mathbb{R}, \omega \in \Omega.$$

By (M4) and (M8),  $F(\omega, \cdot)$  is a distribution function for every  $\omega \in \Omega$ , and thus there exists a probability measure  $\kappa(\omega, \cdot)$  such that  $\kappa(\omega, (-\infty, x]) = F(\omega, x)$  for all  $x \in \mathbb{R}$  and  $\omega \in \Omega$ . For every  $x \in \mathbb{R}$  the function  $F(\omega, x)$  is measurable in  $\omega$ . By a monotone class argument  $\kappa: \Omega \times \mathfrak{B} \rightarrow \overline{\mathbb{R}}_+$  is a probability kernel. Using a monotone class argument based on an almost everywhere interpretation of (M5) and (M7) shows that  $\kappa(\cdot, B) = \Phi(B)$  for all  $B \in \mathfrak{B}$  where  $\mathbf{B}$  denotes here the conditional Borel subset of  $\mathbf{L}^0$  induced by the stable set  $1_\Omega B$ . Every other kernel  $\kappa'$  coincides with  $\kappa$  on the collection of sets  $(-\infty, q]$  for all  $q \in \mathbb{Q}$  which yields uniqueness due to a monotone class argument. The reciprocity identities are immediate from the constructions.  $\square$

#### 4. A CONDITIONAL LEBESGUE INTEGRAL AND A CONDITIONAL VERSION OF FUBINI'S THEOREM

For the sake of simplicity, let  $(\mathbf{X}, \mathcal{X}, \Phi)$  be a finite conditional measure space throughout this section. We introduce a stable indicator. For  $\mathbf{C} = \mathbf{C}|a \in \mathcal{X}$  and  $x \in X$ , define

$$a_x := \vee \{\tilde{a}: \tilde{a} \leq a, \exists \tilde{x} \in C \text{ such that } x|\tilde{a} = \tilde{x}|\tilde{a}\}.$$

Since  $a_{\sum x_i | a_i} = \vee(a_{x_i} \wedge a_i)$  for all concatenations  $\sum x_i | a_i$  in  $X$ , the function  $1_{\mathbf{C}}: X \rightarrow R$  defined by  $1_{\mathbf{C}}(x) := 1|a_x + 0|a_x^c$  is stable, and it is called the *stable indicator of  $\mathbf{C}$* . The following properties can be checked from the definition:

<sup>11</sup>See [6, Section 2].

- (D1)  $1_{\mathbf{C}|a} = 1_{\mathbf{C}}|a + 0|a^c$  for all  $\mathbf{C} \in \mathcal{X}$  and  $a \in \mathcal{A}$ ,
- (D2)  $1_{\sum \mathbf{C}_i|a_i} = \sum 1_{\mathbf{C}_i}|a_i$  for all concatenations  $\sum \mathbf{C}_i|a_i$  in  $\mathcal{X}$ ,
- (D3)  $1_{\sqcup_{n \geq 1} \mathbf{C}_n} = \sum_{n \geq 1} 1_{\mathbf{C}_n}$  for all stable sequences  $(\mathbf{C}_n)$  of conditionally pairwise disjoint elements of  $\mathcal{X}$ ,
- (D4)  $1_{\mathbf{C}} = 1 - 1_{\mathbf{C}^c}$  for all  $\mathbf{C}$  in  $\mathcal{X}$ .

We define a conditionally simple function. A conditionally measurable function  $f: X \rightarrow R_+$  is called *conditionally simple* if there exist a stably finite family  $(x_m)_{1 \leq m \leq n}$  in  $R_+$  and a stably finite family  $(\mathbf{C}_m)_{1 \leq m \leq n}$  of conditionally pairwise disjoint elements in  $\mathcal{X}$  such that  $f = \sum_{1 \leq m \leq n} x_m 1_{\mathbf{C}_m}$ . By a straightforward adaptation of the classical construction, one can show that each conditionally measurable function  $f: X \rightarrow R_+$  is the pointwise limit of a stable increasing sequence of conditionally simple functions. We define the *conditional Lebesgue integral* in three steps. First, for a conditionally simple function  $f = \sum_{1 \leq m \leq n} x_m 1_{\mathbf{C}_m}$ , define

$$\int_X f d\Phi := \sum_{1 \leq m \leq n} x_m \Phi(\mathbf{C}_m).$$

Second, for a conditionally measurable function  $f: X \rightarrow R_+$ , define

$$\int_X f d\Phi := \sup \int_X f_n d\Phi$$

where  $(f_n)$  is a stable increasing sequence converging to  $f$ . Third, for a conditionally measurable function  $f: X \rightarrow R$  define

$$\int_X f d\Phi := \int_X f^+ d\Phi - \int_X f^- d\Phi$$

where  $f^+ = \max\{f, 0\}$  and  $f^- = \max\{-f, 0\}$ , and we call  $f$  *conditionally integrable* whenever  $\int_X |f| d\Phi < \infty$  where  $|f| = f^+ + f^-$ .

We introduce conditional  $\mathbf{L}^p$ -spaces. Let  $f: X \rightarrow R$  be conditionally measurable. For  $1 \leq p < \infty$ , define the *conditional  $p$ -th norm* of  $f$  by

$$\|f\|_p := \left( \int_X |f|^p d\Phi \right)^{1/p}.$$

For  $p = \infty$ , define the *conditional essential supremum norm* of  $f$  by

$$\|f\|_\infty := \inf\{r \geq 0: \Phi(\{|f| > r\}) = 0\}.$$

For arbitrary  $1 \leq p \leq \infty$  with  $p = \tilde{p}|a + \infty|a^c$  where  $\tilde{p} < \infty$ , we denote by  $L^p$  the collection of all conditionally measurable functions  $f: X \rightarrow R$  such that

$$\|f\|_p := \|f\|_{\tilde{p}}|a + \|f\|_\infty|a^c < \infty,$$

where two of them  $f$  and  $\tilde{f}$  are identified whenever  $\Phi(\{f \neq \tilde{f}\}) = 0$  where  $\{f \neq \tilde{f}\} := \mathbf{C}|a$ ,

$$a = \vee \left\{ \tilde{a}: \exists x \in X \text{ such that } f(x)|\tilde{b} \neq \tilde{f}(x)|\tilde{b} \text{ for all } 0 < \tilde{b} \leq \tilde{a} \right\}$$

and

$$C = \left\{ x \in X: f(x)|\tilde{a} \neq \tilde{f}(x)|\tilde{a} \text{ for all } 0 < \tilde{a} \leq a \right\}.$$

It can be checked that each  $\|\cdot\|_p$  ( $1 \leq p \leq \infty$ ) is a stable function which induces a conditional norm on  $\mathbf{L}^p$  in the sense of [6, Definition 5.11]. By following the classical arguments, each  $\mathbf{L}^p$  endowed with the conditional  $p$ -th norm is a conditional Banach space.<sup>12</sup>

As in the classical case, we have a conditional version of the monotone convergence theorem.

<sup>12</sup>A conditional Banach space is defined in the paragraph following [6, Definition 5.11].

**Theorem 4.1.** *Let  $(f_n)$  be an increasing stable sequence of conditionally measurable functions  $f_n: X \rightarrow R_+$ . Then one has*

$$\int \lim f_n d\Phi = \lim \int f_n d\Phi.$$

*Proof.* The proof is a straightforward adaptation of a classical proof.  $\square$

We will prove a conditional version of the Fubini-Tonelli theorem. Let  $(Y, \mathcal{Y}, \Psi)$  be another finite conditional measure space. The *conditional product  $\sigma$ -algebra*  $\mathcal{X} \otimes \mathcal{Y}$  is defined as the conditional  $\sigma$ -algebra generated by

$$\mathcal{E} := \{\mathbf{C} \times \mathbf{D} : \mathbf{C} \in \mathcal{X}, \mathbf{D} \in \mathcal{Y}\}.$$

$\mathcal{E}$  is fixed for the rest of this section.

**Lemma 4.2.** *For  $\mathbf{C} \in \mathcal{X} \otimes \mathcal{Y}$  the function  $\Psi(\mathbf{C}.) : X \rightarrow R$  is conditionally measurable.*

*Proof.* From (S9) and the stability of  $\Psi$  it follows that  $\Psi(\mathbf{C}.)$  is stable. Let

$$\mathcal{D} := \{\mathbf{C} \in \mathcal{X} \otimes \mathcal{Y} : \Psi(\mathbf{C}.) \text{ is conditionally measurable}\}.$$

Since  $\Psi$  is a conditional measure and by (S7), (S10)-(S12),  $\mathcal{D}$  is a conditional Dynkin system. Since for  $\mathbf{C} \times \mathbf{D} \in \mathcal{X} \otimes \mathcal{Y}$  it holds  $\Psi(\mathbf{C} \times \mathbf{D}.) = \Psi(\mathbf{D})1_{\mathbf{C}}$  which is conditionally measurable one has  $\mathcal{E} \subseteq \mathcal{D}$ . The claim follows from a conditional version of the  $\pi$ - $\lambda$  theorem 3.2.  $\square$

**Lemma 4.3.** *There exists a unique finite conditional measure  $\Gamma$  on  $\mathcal{X} \otimes \mathcal{Y}$  such that*

$$\Gamma(\mathbf{C} \times \mathbf{D}) = \Phi(\mathbf{C})\Psi(\mathbf{D})$$

*for all  $\mathbf{C} \in \mathcal{X}, \mathbf{D} \in \mathcal{Y}$ . Moreover, for all  $\mathbf{C} \in \mathcal{X} \otimes \mathcal{Y}$  one has*

$$\Gamma(\mathbf{C}) = \int_Y \Psi(\mathbf{C}_x)\Phi(dx) = \int_X \Phi(\mathbf{C}_y)\Psi(dy).$$

*Proof.* For  $\mathbf{C} \in \mathcal{X} \otimes \mathcal{Y}$  define  $\Gamma(\mathbf{C}) := \int_Y \Psi(\mathbf{C}_x)\Phi(dx)$  which is a stable function by stability of the conditional Lebesgue integral and (S7).  $\Gamma$  satisfies property (M1) by (S8) and the stability of the integral operator, and (M2) by (S11) and Theorem 4.1. For  $\mathbf{C} \in \mathcal{X}, \mathbf{D} \in \mathcal{Y}$  from  $\Psi(\mathbf{C} \times \mathbf{D}.) = \Psi(\mathbf{D})1_{\mathbf{C}}$  one has

$$\Gamma(\mathbf{C} \times \mathbf{D}) = \Psi(\mathbf{C})\Phi(\mathbf{D}). \quad (4.1)$$

Similarly,

$$\tilde{\Gamma}(\mathbf{C}) := \int_Y \Phi(\mathbf{C}_y)\Psi(dy)$$

is a conditional measure on  $\mathcal{X} \otimes \mathcal{Y}$  satisfying (4.1). However by Proposition 3.6 it must hold  $\Gamma = \tilde{\Gamma}$ .  $\square$

For  $f: X \times Y \rightarrow R$  and  $x \in X$  define  $f_x(y) := f(x, y)$  for all  $y \in Y$ , and similarly  $f_y$ .

**Theorem 4.4.** *Let  $f: X \times Y \rightarrow R$  be conditionally integrable. Then we have*

$$\Phi\left(\left(\int_Y |f_x| d\Psi\right)^{-1}(\{\infty\})\right) = \Psi\left(\left(\int_X |f_y| d\Phi\right)^{-1}(\{\infty\})\right) = 0.$$

*Moreover, the stable functions*

$$x \mapsto \int_Y f_x d\Psi \quad \text{and} \quad y \mapsto \int_X f_y d\Phi$$

*are conditionally integrable, respectively, and one has*

$$\int_{X \times Y} f d\Gamma = \int_X \left(\int_Y f_x d\Psi\right) \Phi(dx) = \int_Y \left(\int_X f_y d\Phi\right) \Psi(dy).$$

*Proof.* Suppose  $f = \sum_{1 \leq m \leq n} x_m 1_{\mathbf{C}_m}$ . By (S7) and (S9)-(S12),

$$\mathcal{G} := \{\mathbf{C} \in \mathcal{X} \otimes \mathcal{Y} : \mathbf{C}_x \in \mathcal{Y}\}$$

is a conditional Dynkin system with  $\mathcal{E} \subseteq \mathcal{G}$  for each  $x \in X$ , and thus  $\mathcal{G} = \mathcal{X} \otimes \mathcal{Y}$  by Theorem 3.2. Note that  $f_x = \sum_{1 \leq m \leq n} x_m 1_{(\mathbf{C}_m)_x}$  is conditionally measurable as the stably finite sum of conditionally measurable functions. Then

$$g(x) := \int_Y f_x d\Psi = \sum_{1 \leq m \leq n} x_m \Psi((\mathbf{C}^m)_x)$$

is conditionally measurable by Lemma 4.2. From Lemma 4.3 one obtains

$$\int_X \left( \int_Y f_x d\Psi \right) \Phi(dx) = \sum_{1 \leq m \leq n} x_m \Gamma(\mathbf{C}_m) = \int_{X \times Y} f d\Gamma.$$

For  $f: X \times Y \rightarrow R_+$  the assertions follow from the monotonicity of the conditional Lebesgue integral and Theorem 4.1. The general case follows from the fact that for a conditionally integrable function  $f: X \rightarrow \overline{R}$  one has  $\Phi(|f|^{-1}(\{\infty\})) = 0$ . Indeed, denoting by  $\mathbf{N} := |f|^{-1}(\{\infty\})$ , for  $r \geq 0$  we have  $r 1_{\mathbf{N}} \leq |f|$  which implies  $r\Phi(\mathbf{N}) \leq \int_X |f| d\Phi < \infty$ , and therefore  $\Phi(\mathbf{N}) = 0$ .  $\square$

## 5. A CONDITIONAL VERSION OF THE RADON-NIKODÝM THEOREM

Throughout this section we fix a finite conditional measure space  $(\mathbf{X}, \mathcal{X}, \Phi)$ . A conditional measure  $\Psi$  on  $(\mathbf{X}, \mathcal{X})$  is called *conditionally absolutely continuous* with respect to  $\Phi$  if  $\mathbf{C}$  in  $\mathcal{X}$  and  $\Phi(\mathbf{C}) = \mathbf{0}$  imply  $\Psi(\mathbf{C}) = \mathbf{0}$ . If  $f: X \rightarrow R_+$  is a conditionally measurable function, then from (D1)-(D4) we have that  $\Psi: \mathcal{X} \rightarrow R_+$  defined by

$$\Psi(\mathbf{C}) := \int_X 1_{\mathbf{C}} f d\Phi$$

is a finite conditional measure which is conditionally absolutely continuous with respect to  $\Phi$ . The conditional version of the Radon-Nikodým theorem establishes the following reverse statement.

**Theorem 5.1.** *If  $\Psi$  is a finite conditional measure on  $(\mathbf{X}, \mathcal{X})$  conditionally absolutely continuous with respect to  $\Phi$ , then there is a conditionally measurable function  $f: X \rightarrow R_+$  such that*

$$\Psi(\mathbf{C}) = \int_X 1_{\mathbf{C}} f d\Phi$$

for all  $\mathbf{C} \in \mathcal{X}$ .

The proof is based on the following lemma.

**Lemma 5.2.** *Let  $\Phi_1$  and  $\Phi_2$  be finite conditional measures on  $(\mathbf{X}, \mathcal{X})$  and define  $\Phi_3 := \Phi_2 - \Phi_1$ . Then there exists  $\mathbf{X}_0$  in  $\mathcal{X}$  such that*

- (i)  $\Phi_3(\mathbf{X}) \leq \Phi_3(\mathbf{X}_0)$ ,
- (ii)  $0 \leq \Phi_3(\mathbf{C})$  for all  $\mathbf{C} \in \{\mathbf{C} \in \mathcal{X} : \mathbf{C} \sqsubseteq \mathbf{X}_0\}$ .

*Proof.* Step 1: We show that for all  $r \in R_{++}$  there is a conditional set  $\mathbf{X}_r$  in  $\mathcal{X}$  such that

- (i)'  $\Phi_3(\mathbf{X}) \leq \Phi_3(\mathbf{X}_r)$ ,
- (ii)'  $-r < \Phi_3(\mathbf{C})$  for all  $\mathbf{C} \in \{\mathbf{C} \in \mathcal{X} : \mathbf{C} \sqsubseteq \mathbf{X}_r\}$ .

Let

$$b_0 = \vee \{a : \Phi_3(\mathbf{X})|a \leq 0|a\}.$$

If  $b_0 = 1$ , then  $\mathbf{X}_r = \mathbf{X}|0$  does the required. Assume that  $b_0 < 1$  and let

$$b_1 = \vee \{a : a \leq b_0^c, -r < \Phi_3(\mathbf{C}) \forall \mathbf{C} \in \{\mathbf{C} \in \mathcal{X} : \mathbf{C} \sqsubseteq \mathbf{X}|a\}.$$

If we have  $b_1 = b_0^c$ , then  $\mathbf{X}_r = \mathbf{X}|b_1$  does the required. Suppose that  $b_1 < b_0^c$ . By an exhaustion argument, there exists  $\mathbf{C}_1 \in \{\mathbf{C} \in \mathcal{X} : \mathbf{C} \sqsubseteq \mathbf{X}|(b_0 \vee b_1)^c\}$  such that  $\Phi_3(\mathbf{C}_1)|(b_0 \vee b_1)^c \leq -r|(b_0 \vee b_1)^c$ . We compute

$$\begin{aligned} & \Phi_3(\mathbf{C}_1^\square|(b_0 \vee b_1)^c + \mathbf{X}|b_1 + (\mathbf{X}|0)|b_0) \\ &= \Phi_3(\mathbf{C}_1^\square)|(b_0 \vee b_1)^c + \Phi_3(\mathbf{X})|b_1 + 0|b_0 && \text{by (M1)} \\ &= [\Phi_3(\mathbf{X}) - \Phi_3(\mathbf{C}_1)]|(b_0 \vee b_1)^c + \Phi_3(\mathbf{X})|b_1 + 0|b_0 && \text{by (M5)} \\ &\geq \Phi_3(\mathbf{X})|(b_0 \vee b_1)^c + \Phi_3(\mathbf{X})|b_1 + 0|b_0 \\ &\geq \Phi_3(\mathbf{X}). \end{aligned}$$

If

$$b_2 = \vee\{a : a \leq (b_0 \vee b_1)^c, -r < \Phi_3(\mathbf{C}) \vee \mathbf{C} \in \{\mathbf{C} \in \mathcal{X} : \mathbf{C} \sqsubseteq \mathbf{C}_1^\square|a\}\}$$

is equal to  $(b_0 \vee b_1)^c$ , then  $\mathbf{X}_r = \mathbf{C}_1^\square|(b_0 \vee b_1)^c + \mathbf{X}|b_1 + (\mathbf{X}|0)|b_0$  does the required. Otherwise we have  $b_2 < (b_0 \vee b_1)^c$  and then there exists  $\mathbf{C}_2 \in \{\mathbf{C} \in \mathcal{X} : \mathbf{C} \sqsubseteq \mathbf{C}_1^\square|(b_0 \vee b_1 \vee b_2)^c\}$  such that  $\Phi_3(\mathbf{C}_2)|(b_0 \vee b_1 \vee b_2)^c \leq -r|(b_0 \vee b_1 \vee b_2)^c$  by an exhaustion argument. Since  $\mathbf{C}_1 \sqcap \mathbf{C}_2 = \mathbf{X}|0$ , one has similarly to the previous computation

$$\Phi_3((\mathbf{C}_1 \sqcup \mathbf{C}_2)^\square|(b_0 \vee b_1 \vee b_2)^c + \mathbf{C}_1^\square|b_2 + \mathbf{X}|b_1 + (\mathbf{X}|0)|b_0) \geq \Phi_3(\mathbf{X}),$$

and we can repeat the previous reasoning. If our procedure does not yield the required after finitely many steps, we obtain a sequence  $(b_n)_{n \geq 0}$  of pairwise disjoint elements in  $\mathcal{A}$  and a sequence  $(\mathbf{C}_n)_{n \geq 0}$  of pairwise disjoint conditional sets in  $\mathcal{X}$  such that

$$\begin{aligned} & \Phi_3((\mathbf{C}_1 \sqcup \dots \sqcup \mathbf{C}_n)^\square|(b_0 \vee b_1 \vee \dots \vee b_n)^c \\ &+ (\mathbf{C}_1 \sqcup \dots \sqcup \mathbf{C}_{n-1})^\square|b_n + \dots + \mathbf{C}_1^\square|b_2 + \mathbf{X}|b_1 + (\mathbf{X}|0)|b_0) \geq \Phi_3(\mathbf{X}) \end{aligned}$$

and

$$\Phi_3(\mathbf{C}_n)|(b_0 \vee b_1 \vee \dots \vee b_n)^c \leq -r|(b_0 \vee b_1 \vee \dots \vee b_n)^c$$

hold for all  $n \geq 0$ . If  $b = \wedge_{n \geq 0} b_n^c > 0$ , then (M2) yields<sup>13</sup>

$$\Phi_3(\sqcup \mathbf{C}_n|b) = \sum_{n \geq 1} \Phi_3(\mathbf{C}_n)|b + 0|b^c = -\infty|b + 0|b^c$$

which contradicts the finiteness of the conditional measures  $\Phi_1$  and  $\Phi_2$ . Thus  $(b_n)_{n \geq 0} \in p(1)$  and  $\mathbf{X}_r := \sum \mathbf{D}_n|b_n$  where  $\mathbf{D}_0 := \mathbf{X}|0$ ,  $\mathbf{D}_1 := \mathbf{X}$  and  $\mathbf{D}_n := (\mathbf{C}_1 \sqcup \dots \sqcup \mathbf{C}_{n-1})^\square$ ,  $n \geq 2$ , satisfies the required.

Step 2: By Step 1, we may choose  $\mathbf{X}_{1/n}$  such that  $\Phi_3(\mathbf{X}) \leq \Phi_3(\mathbf{X}_{1/n})$ ,  $-1/n < \Phi_3(\mathbf{C})$  for all  $\mathbf{C} \in \{\mathbf{C} \in \mathcal{X} : \mathbf{C} \sqsubseteq \mathbf{X}_{1/n}\}$ , and  $\mathbf{X}_{1/(n+1)} \sqsubseteq \mathbf{X}_{1/n}$  for all  $n \in \mathbb{N}_s$ . Then  $\mathbf{X}_0 := \sqcap \mathbf{X}_{1/n}$  yields the claim by (M9).

□

The proof of Theorem 5.1 follows by an adaptation of classical arguments.

*Proof.* Let  $\mathcal{G}$  be the collection of all conditionally measurable functions  $f : X \rightarrow R_+$  such that

$$\int_X 1_{\mathbf{C}} f d\Phi \leq \Psi(\mathbf{C})$$

for all  $\mathbf{C} \in \mathcal{X}$ . Inspection shows that  $\mathcal{G}$  is a stable collection of stable functions such that  $\max\{f, g\} \in \mathcal{G}$  for all  $f, g \in \mathcal{G}$ . Let  $r = \sup_{f \in \mathcal{G}} \int_X f d\Phi < \infty$ . There exists a stable sequence  $(g_n)$  in  $\mathcal{G}$  such that  $\lim \int_X g_n d\Phi = r$ . Set

$$f_n = \max_{1 \leq m \leq n} g_m := \sum \left( \max_{1 \leq m \leq n_i} g_m \right) | a_i$$

<sup>13</sup>The limit of the series  $\sum_{n \geq 1} \Phi_3(\mathbf{C}_n)$  is the limit of the corresponding stable series  $\sum_{n \geq 1} \Phi_3(\mathbf{C}_n)$  where for  $n = \sum n_i | a_i$  we put  $\mathbf{C}_n = \sum \mathbf{C}_{n_i} | a_i$ .

for each  $n = \sum n_i | a_i \in \mathbb{N}_s$ . By the monotone convergence theorem 4.1,  $\int_X f d\Phi = r$  for  $f = \sup f_n$  which is an element of  $\mathcal{G}$ . Thus we have

$$\tilde{\Psi}(\mathbf{C}) := \int_X 1_{\mathbf{C}} f d\Phi \leq \Psi(\mathbf{C})$$

for all  $\mathbf{C} \in \mathcal{X}$ . It remains to show that  $\Gamma := \Psi - \tilde{\Psi} \equiv 0$ . For the sake of contradiction, suppose that

$$b = \vee \{a : \Gamma(\mathbf{X})|a > 0|a\} > 0,$$

and we may assume that  $b = 1$  which implies  $\Psi(\mathbf{X}) > 0$  since  $\Gamma$  is conditionally absolutely continuous with respect to  $\Phi$ . Set  $s := \Gamma(\mathbf{X})/2\Phi(\mathbf{X})$ . By Lemma 5.2, applied to  $\Phi_2 = \Gamma$  and  $\Phi_1 = s\Phi$ , there exists  $\mathbf{X}_0 \in \mathcal{X}$  such that  $\Gamma(\mathbf{X}_0) - s\Phi(\mathbf{X}_0) \geq \Gamma(\mathbf{X}) - s\Phi(\mathbf{X}) > 0$  and  $\Gamma(\mathbf{C}) \geq s\Phi(\mathbf{C})$  for all  $\mathbf{C} \in \{\mathbf{C} \in \mathcal{X} : \mathbf{C} \subseteq \mathbf{X}_0\}$ . Then  $\tilde{f} = f + s1_{\mathbf{X}_0}$  is an element of  $\mathcal{G}$  with  $\int_X \tilde{f} d\Phi > r$  which is contradictory.  $\square$

## 6. A CONDITIONAL VERSION OF THE DANIELL-STONE THEOREM AND RIESZ REPRESENTATION THEOREM

In this section we will prove a conditional version of the Daniell-Stone theorem thanks to which conditional versions of the Riesz representation theorem on the conditionally  $n$ -dimensional Euclidean space are established.

**Definition 6.1.** Given a conditional set  $\mathbf{X}$ , a stable collection  $\mathcal{L}$  of stable functions  $f : X \rightarrow R$  is called a *conditional Stone vector lattice* whenever  $f + g, rf$  and  $\min\{f, 1\}$  are elements of  $\mathcal{L}$  for all  $f, g \in \mathcal{L}$  and  $r \in R$  and there exist  $f \in \mathcal{L}$  and  $x \in X$  such that  $f(x)|a \neq 0|a$  for all  $a > 0$ .

**Theorem 6.2.** Let  $\mathcal{L}$  be a conditional Stone vector lattice and  $L : \mathcal{L} \rightarrow R$  stable, linear and such that  $L(f) \geq 0$  whenever  $f \geq 0$  and  $L(f_n) \downarrow 0$  whenever  $f_n \downarrow 0$ . Then there exists a conditional measure  $\Phi$  on  $\Sigma(\mathcal{L})$  such that  $L(f) = \int_X f d\Phi$  for all  $f$  in  $\mathcal{L}$ .

*Proof.* For  $f, g$  in  $\mathcal{L}$  with  $f \leq g$  we define  $[\mathbf{f}, \mathbf{g}] := \mathbf{C}|a$  where

$$a = \vee \{\tilde{a} : \exists x \in X \text{ such that } f(x)|\tilde{a} < g(x)|\tilde{a}\}$$

and

$$C = \{(x, r) \in X \times R : f(x)|a \leq r|a < g(x)|a\}.$$

The collection  $\mathcal{X}$  of all conditional unions of stably finite families  $([\mathbf{f}_m, \mathbf{g}_m])_{1 \leq m \leq n}$  of pairwise disjoint elements is a conditional ring on  $\mathbf{X} \times \mathbf{R}$ . The stable function  $\Psi : \mathcal{X} \rightarrow \overline{R}_+$  given by

$$\Psi(\sqcup_{1 \leq m \leq n} [\mathbf{f}_m, \mathbf{g}_m]) := \sum_{1 \leq m \leq n} L(g_m - f_m)$$

is a conditional pre-measure which by Theorem 3.5 extends to a conditional measure on  $\Sigma(\mathcal{X})$ . By inspection we have  $\mathcal{M} := \Sigma(\{f^{-1}(\cdot]1, \infty[) : f \in \mathcal{L}\}) = \Sigma(\mathcal{L})$ . For  $f \in \mathcal{L}$  let  $f^{-1}(\cdot]1, \infty[)$  be of the form  $\mathbf{D}|d$ , and for  $x \in X$  and  $n \in \mathbb{N}_s$  set

$$a_x = \vee \{\tilde{a} : f(x)|\tilde{a} \leq 1|\tilde{a}\}, \quad b_x = \vee \left\{ \tilde{b} : 1|\tilde{b} < f(x)|\tilde{b} < \frac{n+1}{n}|\tilde{b} \right\}, \quad c_x = \vee \left\{ \tilde{c} : f(x)|\tilde{c} \geq \frac{n+1}{n}|\tilde{c} \right\},$$

and  $g_n(x) := 0|(a_x \vee d) + n(f(x) - 1)|b_x + 1|c_x$ . Since  $\sqcup[\mathbf{0}, \mathbf{g}_n] = f^{-1}(\cdot]1, \infty[) \times [\mathbf{0}, \mathbf{1}[$  the stable function  $\Phi(\mathbf{C}) := \Psi(\mathbf{C} \times [\mathbf{0}, \mathbf{1}[)$  is a conditional measure on  $\mathcal{M}$ . The representation  $L(f) = \int_X f d\Phi$  for all  $f \in \mathcal{L}$  follows from (M8) and Theorem 4.1.  $\square$

We will need a conditional version of Dini's lemma:

**Lemma 6.3.** *Let  $(\mathbf{X}, \mathcal{T})$  be a conditionally compact topological space<sup>14</sup> and  $(\mathbf{f}_n)$  a conditionally decreasing conditional sequence of conditionally continuous functions<sup>15</sup> conditionally converging pointwisely to a conditionally continuous function  $\mathbf{f}$ . Then for all  $\mathbf{r} > \mathbf{0}$  there exists  $\mathbf{n}_0$  in  $\mathbf{N}_s$  such that  $\sup_{\mathbf{x} \text{ in } \mathbf{X}} |\mathbf{f}_n(\mathbf{x}) - \mathbf{f}(\mathbf{x})| \leq \mathbf{r}$  for all  $\mathbf{n} \geq \mathbf{n}_0$ .*

*Proof.* The proof is similar to the classical proof by using the definition of conditional compactness.  $\square$

For  $d \in \mathbf{N}_s$  let  $\mathcal{C}$  denote the stable collection of all stable continuous functions  $f : R^d \rightarrow R$ .<sup>16</sup>  $f \in \mathcal{C}$  has *conditionally compact support* whenever  $\mathbf{cl}(f^{-1}(\{\mathbf{0}\}^\square))$  is conditionally compact where  $\mathbf{cl}$  denotes the conditional closure.<sup>17</sup> We denote by  $\mathcal{C}_c$  the stable sub-collection of  $\mathcal{C}$  of all conditionally compactly supported functions. Both  $\mathcal{C}$  and  $\mathcal{C}_c$  are conditional Stone vector lattices.

We denote by

$$d(x, y) = |x - y| := \sqrt{\sum_{1 \leq m \leq n} (y_m - x_m)^2}$$

the stable metric on  $R^n$  which induces the conditional Euclidean metric. A finite conditional measure  $\Phi$  on the conditional Borel  $\sigma$ -algebra  $\mathcal{B}^n$  is called *conditionally tight* whenever

$$\Phi(\mathbf{C}) = \sup\{\Phi(\mathbf{D}) : \mathbf{D} \subseteq \mathbf{C} \text{ conditionally compact}\}$$

for all  $\mathbf{C} \in \mathcal{B}^n$ .

**Corollary 6.4.** *Let  $L : \mathcal{C} \rightarrow R$  be stable, linear and such that  $L(f) \geq 0$  whenever  $f \geq 0$ . Then there exists a finite conditionally tight measure  $\Phi$  on  $\mathcal{B}^n$  such that  $L(f) = \int_{R^n} f d\Phi$  for all  $f \in \mathcal{C}$ .*

*Proof.* Let  $(f_l)$  be a stable sequence in  $\mathcal{C}$  with  $f_l \downarrow 0$ . Let  $C_k = \{x \in R^n : |x| \leq k\}$ ,  $k \in \mathbf{N}_s$ . Note that  $\mathbf{C}_k$  is conditionally compact by [6, Theorem 4.6]. Put  $g_k = \max\{1 - d(\cdot, C_k), 0\}$  and  $h_{kl} = g_k f_l + (1 - g_k) f_1 \cdot | \cdot | / 2k$ . One has  $f_l \leq h_{lk}$  for all  $l, k \in \mathbf{N}_s$ . Fix  $r > 0$ , and choose  $k$  large enough such that  $1/(2k)L(f_1(1 - g_1)| \cdot |) < r/2$ , and afterwards choose  $l$  sufficiently large such that  $L(g_k f_l) < r/2$  by Lemma 6.3. We have  $L(f_l) \leq L(g_k f_l) + 1/(2k)L((1 - g_1)f_1| \cdot |) < r$ . By Theorem 6.2 there exists a finite conditional measure  $\Phi$  on  $\mathcal{B}^n$  representing  $L$ . The regularity condition follows from an adaptation of the arguments in the proof of [3, Proposition 1.5].  $\square$

**Corollary 6.5.** *Let  $L : \mathcal{C}_c \rightarrow R$  be stable, linear and such that  $L(f) \geq 0$  whenever  $f \geq 0$ . Then there exists a conditional measure  $\Phi$  on  $\mathcal{B}^n$  such that  $L(f) = \int_{R^n} f d\Phi$  for all  $f \in \mathcal{C}_c$ . Moreover, one has  $\Phi(\mathbf{K}) < \infty$  for all conditionally compact intervals  $\mathbf{K}$  and  $\Phi(\mathbf{C}) = \sup\{\Phi(\mathbf{D}) : \mathbf{D} \subseteq \mathbf{C} \text{ conditionally compact}\}$  for all  $\mathbf{C} \in \mathcal{B}^n$  with  $\Phi(\mathbf{C}) < \infty$ .*

*Proof.* In order to obtain the assumptions of Theorem 6.2 apply Lemma 6.3 to the stable sequence  $(1_{\mathbf{K}} f_n)$  where  $\mathbf{K}$  denotes the support of  $f_1$ . For a conditionally compact interval  $\mathbf{K}$  and  $f = \max\{1 - d(\cdot, K), 0\}$  one has  $\Phi(\mathbf{K}) = \int_{R^n} f d\Phi \leq L(f)$ . The conditional regularity condition follows similarly to Corollary 6.4.  $\square$

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<sup>14</sup>See [6, Definition 3.24].

<sup>15</sup>See [6, Definition 3.8].

<sup>16</sup>See [6, Proposition 3.11] for the relation between classical and conditional continuity.

<sup>17</sup>See [6, Definition 3.3].

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